Published by IOP Publishing for SISSA

Received: *September 26, 2008*
Revised: *October 21, 2008*
Accepted: *October 23, 2008*
Published: *November 6, 2008*

# Topological twisting of multiple M2-brane theory

**Kanghoon Lee**

*Department of Physics, College of Science, Yonsei University, Shinchon-dong, Seodaemun-gu, Seoul 120-749, Korea*
*E-mail:* lkh@phya.yonsei.ac.kr

**Sangmin Lee**

*Department of Physics and Astronomy, Seoul National University, Gwanakro Sillim-dong, Gwanak-gu, Seoul 151-747, Korea*
*E-mail:* sangmin@snu.ac.kr

**Jeong-Hyuck Park**

*Department of Physics, Sogang University, Shinsu-dong, Mapo-gu, Seoul 121-742, Korea*
*E-mail:* park@sogang.ac.kr

Abstract: Bagger-Lambert-Gustavsson theory with infinite dimensional gauge group has been suggested to describe M5-brane as a condensation of multiple M2-branes. Here we perform a topological twisting of the Bagger-Lambert-Gustavsson theory. The original SO(8) $R$-symmetry is broken to SO(3) $\times$ SO(5), where the former may be interpreted as a diagonal subgroup of the Euclidean M5-brane world-volume symmetry SO(6), while the latter is the isometry of the transverse five directions. Accordingly the resulting action contains an one-form and five scalars as for the bosonic dynamical fields. We further lift the action to a generic curved three manifold. In order to make sure the genuine topological invariance, we construct an off-shell supersymmetric formalism such that the scalar supersymmetry transformations are nilpotent strictly off-shell and independent of the metric of the three manifold. The one loop partition function around a trivial background yields the Ray-Singer torsion. The BPS equation involves an M2-brane charge density given by a Nambu-Goto action defined in an internal three-manifold.

Keywords: Brane Dynamics in Gauge Theories, Topological Field Theories, M-Theory.

## Contents





## 1. Introduction

D-branes have played a crucial role in understanding non-perturbative dynamics of string theory. The M2 and M5 branes are expected to play a similar role in $\mathcal{M}$-theory, but due to their intrinsically non-perturbative nature, their world-volume theories remain much less understood than those of D-branes. In particular, a Lorentz invariant Lagrangian descriptions of the interacting conformal field theories living in the M2/M5 world-volume have been missing.

Being the only branes in $\mathcal{M}$-theory (in flat eleven-dimensions), M2 and M5 branes are intricately related. First, they are electromagnetic dual to each other with respect to the four-form field strength $G_{(4)} = \mathrm{d}C_{(3)}$ in the eleven-dimensional supergravity. Second, M2-branes can end on M5-branes just as fundamental strings end on D-branes. Roughly speaking, quantum excitations of open M2-branes should give rise to a microscopic formulation of the M5-brane world-volume theory. Third, the self-dual three-form flux $H_{(3)}$ on M5-branes carries M2-brane charge. Finally, M2-branes in a background $G_{(7)} = *G_{(4)}$ can be blown up to M5-branes by an $\mathcal{M}$-theory version of the Myers' effect [1].

Some time ago, Basu and Harvey [2] studied the BPS configuration of M2-branes ending on M5-branes, which exhibits many of these relations at once.[1] In analogy with the D1-D3 interpretation of Nahm's equations [3, 4], they argued that the "non-Abelian" M2-brane world-volume theory should admit a sort of fuzzy three-sphere solution [5]. Around the same time, from a different perspective, Schwarz [6] raised the possibility of using superconformal Chern-Simons theories as for the description of the M2-brane dynamics.

Inspired by these pioneering works, Bagger-Lambert [7] and Gustavsson [8] (BLG) succeeded in writing down an $\mathcal{N} = 8$ superconformal Chern-Simons-matter theory with SO(8) $R$-symmetry. The BLG Lagrangian was interpreted as the low energy of limit of the world-volume theory of two M2-branes in a certain M-theory background [9, 10].

The action is based on a gauge symmetry generated by the so-called three-algebra. As for a conventional, ghost-free field theory with a *finite* number of fields, the BLG theory admits only one gauge group $\mathrm{SO}(4) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2)$ with opposite levels for the two Chern-Simons terms, and matter fields come in bi-fundamental representations. The uniqueness is due to the surprisingly strong constraint imposed by the three-algebra structure. In order to free this severe restriction, one can consider either Lorentzian gauge groups [11] or infinite dimensional three-algebras. The latter can be realized as a volume-preserving diffeomorphism of an auxiliary three-dimensional manifold. Combining both the original and the auxiliary three manifolds leads to a six-dimensional manifold, and the BLG theory with an infinite-dimensional gauge group may have a natural origin as an M5-brane action [12 − 14]. In particular, in ref. [13] it has been shown that by generalizing the Brink-Di Vecchia-Howe-Polyakov method, Nambu-Goto action for a $p$-brane can be reformulated as a $d$-dimensional gauged nonlinear sigma model having a Nambu $(p+1-d)$-bracket squared potential. While the choice $d = p - 1$ leads to the Yang-Mills potential, the choice $d = p - 2$ leads to the Nambu three-bracket potential, and hence an infinite dimensional three-algebra. In particular, an M5-brane may be described by a condensation of M2-branes.

The connection between multiple M2's and an M5 motivates us to twist the (Euclidean) BLG theory by diagonalizing the SO(3) Lorentz symmetry and an SO(3) subgroup of the SO(8) $R$-symmetry. The resulting action will contain five scalars which can be viewed as the physical degrees of freedom along the five transverse directions of an M5-brane. While the twisting we perform works for any three-algebra, as an application, we will consider infinite dimensional gauge group or volume-preserving diffeomorphism in an internal three manifold at the end of the paper.

A quantum field theory is called topological if all vacuum expectation values (vevs) of a certain set of operators (observables) are metric-independent. In particular, topological quantum field theories (TQFTs) of cohomological type are constructed as follows. Let us assume there is a nilpotent symmetry of the action $Q$, such that $Q^2 = 0$. It follows that, at least formally, one can deform the Lagrangian by adding an arbitrary $Q$-exact term without affecting the partition function or the vevs of observables (which are defined as elements in the cohomology of $Q$). Since $Q$ is a symmetry of the action, the Lagrangian can be expressed as a sum of a $Q$-exact and a $Q$-closed piece. The theory is therefore independent

[1] See ref. [15] for further discussion.



JHEP11(2008)014

of any coupling constant in the $Q$-exact piece. Moreover, if the energy momentum tensor is $Q$-exact all vevs of observables are metric-independent and the theory is topological.

The organization of the present paper is as follows.

In section 2, we construct the twisted BLG theory. We begin with writing down the Euclidean version of the BLG theory. Then, we perform a twist which preserves an $\mathrm{SO}(3) \times \mathrm{SO}(5) \subset \mathrm{SO}(8)$ $R$-symmetry. On-shell nilpotency of the scalar supersymmetries and the corresponding BPS equations are also presented.

In section 3, to make sure the genuine topological invariance, we introduce some auxiliary fields such that the supersymmetry algebra closes strictly off-shell and the supersymmetry transformations are independent of the three-manifold metric. Using the off-shell supersymmetric formulation, we separate the twisted BLG action into a $Q$-closed topological part and a $Q$-exact part, thereby verifying the topological invariance of the theory.

In section 4, we initiate the study of observables of the theory. We explicitly derive those observables which can be obtained from the Lagrangian through a descent relation. Then, we explore the possibility of a Wilson-loop operator, but our analysis indicates that the twisted BLG theory does not admit a $Q$-closed Wilson loop operator. Then we take a first step toward the perturbative computation of the partition function. The one loop determinants around a trivial background turns out to be the Ray-Singer torsion.

In section 5, we interpret our results from the M5-brane point of view. Realizing infinite dimensional gauge symmetry as volume preserving diffeomorphism in an internal three manifold, our twisted theory can be viewed as partial topological twisting of a six-dimensional theory, where the six-dimensional space has the fiber bundle structure: at each point in a three manifold (base), there exists a corresponding internal three manifold (fiber). The BPS equations then involves an M2-brane charge density given by a Nambu-Goto action defined in an internal three-manifold.

In section 6, we conclude with some comments on future work.

Appendix carries some relevant useful identities.

## 2. Twisted Bagger-Lambert-Gustavsson theory

### 2.1 Euclidean Bagger-Lambert-Gustavsson theory

To start, we present the Euclidean version of the Bagger-Lambert-Gustavsson Lagrangian:

$$\begin{aligned}\mathcal{L}_{\text{Euclidean}} &= i\epsilon^{\mu\nu\lambda}\left(\frac{1}{2}f^{abcd}A_{\mu ab}\partial_\nu A_{\lambda cd} - \frac{1}{3}f^{cdag}f^{efb}{}_g A_{\mu ab}A_{\nu cd}A_{\lambda ef}\right)\\ &\quad + \mathrm{Tr}\left[\frac{1}{2}(D_\mu X^I)^2 - \frac{i}{2}\bar{\Psi}\Gamma^\mu D_\mu\Psi + \frac{i}{4}\bar{\Psi}\Gamma_{IJ}[X^I, X^J, \Psi] + \frac{1}{12}[X^I, X^J, X^K]^2\right]. \end{aligned} \tag{2.1}$$

There are some common as well as distinct features compared to the original Minkowskian case [7]. In terms of an explicit basis of the three-algebra,

$$[T^a, T^b, T^c] = f^{abc}{}_d T^d\,, \tag{2.2}$$

the dynamical variables take values in the three-algebra, e.g. $X^I = X^I_a T^a$. The trace is always taken over second-order in three-algebra variables such that, in fact it involves a metric which can raise or low the gauge index. The covariant derivatives are the same as in the Minkowskian case [7]:

$$D_\mu X^I_a = \partial_\mu X^I_a + \tilde{A}_{\mu a}{}^b X^I_b\,, \qquad D_\mu X^{aI} = \partial_\mu X^{aI} - X^{bI}\tilde{A}_{\mu b}{}^a = \partial_\mu X^{aI} + \tilde{A}_\mu{}^a{}_b X^{bI}\,. \tag{2.3}$$

The tilde symbol denotes the contraction with the structure constant of the three-algebra,

$$\tilde{A}_{\mu a}{}^b := A_{\mu cd} f^{cd}{}_a{}^b\,. \tag{2.4}$$

The gauge symmetry is then realized by

$$\delta_\Lambda X^I_a = -\tilde{\Lambda}_a{}^b X^I_b\,, \quad \delta_\Lambda \Psi_a = -\tilde{\Lambda}_a{}^b \Psi_b\,, \quad \delta_\Lambda A_{\mu ab} = D_\mu \Lambda_{ab} = \partial_\mu \Lambda_{ab} + \tilde{A}_{\mu a}{}^c \Lambda_{cb} + \tilde{A}_{\mu b}{}^c \Lambda_{ac}\,. \tag{2.5}$$

The key difference, compared to the Minkowskian signature [7], is that the Euclidean action contains only the 'holomorphic' part of the spinor such that $\bar{\Psi}$ is defined to be the *charge conjugation* of $\Psi$:

$$\bar{\Psi} := \Psi^T \mathcal{C}\,. \tag{2.6}$$

This is due to the fact that the three-dimensional Euclidean space does not admit real spinors i.e. Majorana condition. Here $\mathcal{C}$ is the charge conjugation matrix in eleven dimensions satisfying

$$\mathcal{C}\Gamma^M \mathcal{C}^{-1} = -(\Gamma^M)^T\,, \qquad \mathcal{C}^T = -\mathcal{C}\,, \tag{2.7}$$

where $M$ is the eleven-dimensional vector index which decomposes into $\mu = 1, 2, 3$ and $I = 4, 5, \ldots, 11$. Throughout the paper, the complex conjugation of spinors will never appear as we focus on the Euclidean space. Further the dynamical spinor field $\Psi$ has a definite chirality over $(1, 2, 3)$-space:

$$\Gamma^{123}\Psi = +i\Psi\,. \tag{2.8}$$

In our convention, the field strength is defined by

$$\tilde{F}_{\mu\nu}{}^a{}_b = \partial_\mu \tilde{A}_\nu{}^a{}_b - \partial_\nu \tilde{A}_\mu{}^a{}_b + \tilde{A}_\mu{}^a{}_c \tilde{A}_\nu{}^c{}_b - \tilde{A}_\nu{}^a{}_c \tilde{A}_\mu{}^c{}_b\,, \tag{2.9}$$

of which the overall sign is opposite to the original convention by Bagger and Lambert [7] but faithful to the standard convention.

Last but not least, the Euclidean action (2.1) is invariant under the following sixteen supersymmetry transformation:[2]

$$\begin{aligned} \delta X^I_a &= i\bar{\mathcal{E}}\Gamma^I \Psi_a\,, \\ \delta \Psi_a &= D_\mu X^I_a \Gamma^\mu \Gamma_I \mathcal{E} - \frac{1}{6} X^I_b X^J_c X^K_d f^{bcd}{}_a \Gamma^{IJK}\mathcal{E}\,, \\ \delta \tilde{A}_{\mu ab} &= i\bar{\mathcal{E}}\Gamma_\mu \Gamma_I X^I_c \Psi_d f^{cd}{}_{ab}\,, \end{aligned} \tag{2.10}$$

which take exactly the same form as in the Minkowskian case. The supersymmetry parameter $\mathcal{E}$ possesses the opposite chirality compared to (2.8),

$$\Gamma^{123}\mathcal{E} = -i\mathcal{E}\,. \tag{2.11}$$

[2] In addition to the *ordinary* supersymmetry (2.11), the Euclidean action enjoys sixteen *conformal* supersymmetry [20], which can be also twisted to define a novel topological theory on an arbitrary three-dimensional cone, as was done for $\mathcal{N} = 4$ super Yang-Mills defined on a four-dimensional cone [21].

### 2.2 Description of the twist

We now come to the description of the twist we perform. Under Spin(11) $\to$ Spin(3) $\times$ Spin(3) $\times$ Spin(5), the eleven-dimensional gamma matrices can be decomposed as

$$\Gamma^{\mu} = \sigma^{\mu} \otimes 1 \otimes 1 \otimes \sigma^{3}\,, \qquad \Gamma^{\mu+3} = 1 \otimes \sigma^{\mu} \otimes 1 \otimes \sigma^{1}\,, \qquad \Gamma^{i+6} = 1 \otimes 1 \otimes \gamma^{i} \otimes \sigma^{2}\,, \tag{2.12}$$

where $\sigma^{\mu}$, $\mu = 1, 2, 3$ are $2 \times 2$ Pauli matrices

$$\sigma^{1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^{2} = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}, \qquad \sigma^{3} = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.13}$$

and $\gamma^{i}$, $i = 1, 2, \ldots, 5$ are $4 \times 4$ gamma matrices in Euclidean five dimensions, satisfying

$$\gamma^{i}\gamma^{j} + \gamma^{j}\gamma^{i} = 2\delta^{ij}\,, \qquad \gamma^{12345} = 1\,. \tag{2.14}$$

The charge conjugation matrix in (2.7) takes the explicit form:

$$\mathcal{C} = \epsilon \otimes \epsilon \otimes C \otimes 1\,, \tag{2.15}$$

where $\epsilon = i\sigma^{2}$ as usual, and $C$ is the five-dimensional charge conjugation matrix,

$$\epsilon\sigma^{\mu}\epsilon^{-1} = -(\sigma^{\mu})^{T}\,, \qquad C\gamma^{i}C^{-1} = +(\gamma^{i})^{T}\,, \qquad C^{T} = -C\,. \tag{2.16}$$

The so(8) chiral matrix is then

$$\Gamma^{123} = -i\Gamma^{456\cdots 11} = 1 \otimes 1 \otimes 1 \otimes i\sigma^{3}\,. \tag{2.17}$$

Consequently the eleven-dimensional spinors carry four indices $\Psi^{\dot{\alpha}\dot{\beta}\alpha\pm}$. The first two $\dot{\alpha}$, $\dot{\beta}$ indices are for the so(3) spinor indices and the third one $\alpha$ is for so(5) spinor indices running from one to four. The last one $\pm$ denotes the so(8) chirality. Since the dynamical spinor carries the definite chirality (2.8) we have $\Psi^{\dot{\alpha}\dot{\beta}\alpha-}= 0$. Similarly from (2.11), we have for the supersymmetry parameter $\mathcal{E}^{\dot{\alpha}\dot{\beta}\alpha+}= 0$. The twist we focus on in the present paper amounts to replacing the three-dimensional rotation group by the diagonal subgroup of Spin(3) $\times$ Spin(3). Accordingly, the twisted spinors admit the following expansion:

$$\Psi^{\dot{\alpha}\dot{\beta}\alpha+} = \frac{1}{\sqrt{2}}\left(i\eta^{\alpha}\epsilon^{\dot{\alpha}\dot{\beta}} + \chi^{\alpha}_{\mu}(\sigma^{\mu}\epsilon)^{\dot{\alpha}\dot{\beta}}\right), \qquad \mathcal{E}^{\dot{\alpha}\dot{\beta}\alpha-} = \frac{1}{\sqrt{2}}\left(i\varepsilon^{\alpha}\epsilon^{\dot{\alpha}\dot{\beta}} + \varepsilon^{\alpha}_{\mu}(\sigma^{\mu}\epsilon)^{\dot{\alpha}\dot{\beta}}\right). \tag{2.18}$$

Namely the fermions decompose into a SO(3) scalar $\eta$, $\varepsilon$ and a one-form $\chi_{\mu}\mathrm{d}x^{\mu}$, $\varepsilon_{\mu}\mathrm{d}x^{\mu}$. In analogue to (2.6), we also define the charge conjugation of the SO(5) spinors for convenience:

$$\bar{\eta} = \eta^{T}C\,, \qquad \bar{\chi}_{\mu} = \chi^{T}_{\mu}C\,. \tag{2.19}$$

Finally for bosons, our twist prescribes to decompose the eight bosonic fields into a SO(3) one-form and five scalars:

$$X^{I} \longrightarrow \left(\, X_{\mu}\mathrm{d}x^{\mu}\,,\, Y^{i}\,\right)\,. \tag{2.20}$$

### 2.3 Twisted Lagrangian

Taking the decompositions (2.12), (2.18), (2.20) and an identity (A.3) into account, it is straightforward to rewrite the Euclidean Bagger-Lambert-Gustavsson action (2.1) in terms of the anti-commuting fields $\eta, \chi_\mu$ and the bosons $A_\mu, X_\mu, Y^i$. The resulting action defines our twisted Bagger-Lambert-Gustavsson theory in three-dimensions:

$$\mathcal{S}_{\text{twisted}} = \int \mathrm{d}^3x\ \mathcal{L}_{\text{twisted}}\,, \qquad \mathcal{L}_{\text{twisted}} = \mathcal{L}_{\text{top}} + \sqrt{g}L_g\,, \tag{2.21}$$

where

$$\begin{aligned}\mathcal{L}_{\text{top}} &= i\epsilon^{\mu\nu\lambda}\left(\frac{1}{2}A^{+}_{\mu ab}\partial_\nu\tilde{A}^{+ab}_{\lambda} + \frac{1}{3}A^{+}_{\mu ab}\tilde{A}^{+a}_{\nu}{}_{c}\tilde{A}^{+cb}_{\lambda}\right)\\ &\quad -\epsilon^{\mu\nu\lambda}\mathrm{Tr}\left[\frac{1}{2}\bar{\chi}_\mu D^{+}_{\nu}\chi_\lambda - i\frac{1}{2}\bar{\chi}_\mu\gamma^i\left[\chi_\nu, X_\lambda, Y_i\right]\right]\end{aligned} \tag{2.22}$$

and

$$\begin{aligned}L_g = \mathrm{Tr}\Bigg[&\frac{1}{4}(D_\mu X_\nu - D_\nu X_\mu)(D^\mu X^\nu - D^\nu X^\mu) + \frac{1}{2}\left(D_\mu X^\mu + i\frac{1}{6\sqrt{g}}\epsilon^{\mu\nu\lambda}[X_\mu, X_\nu, X_\lambda]\right)^2\\ &+\frac{1}{2}D^{+}_{\mu}Y^iD^{-\mu}Y_i + \frac{1}{12}[Y^i, Y^j, Y^k][Y_i, Y_j, Y_k] + \frac{1}{4}[X_\mu, Y^j, Y^k][X^\mu, Y_j, Y_k]\\ &+\bar{\eta}D^{-}_{\mu}\chi^\mu + i\bar{\eta}\gamma^i\left[Y_i, X_\mu, \chi^\mu\right] + i\frac{1}{4}\bar{\eta}\gamma^{ij}\left[Y_i, Y_j, \eta\right] + i\frac{1}{4}\bar{\chi}^\mu\gamma^{ij}\left[Y_i, Y_j, \chi_\mu\right]\Bigg]\,.\end{aligned} \tag{2.23}$$

In the above, we have coupled the action to a generic three-dimensional metric $g_{\mu\nu}$, such that all the derivatives are covariant with respect to both *diffeomorphisms* and *gauge transformations*, and that $\epsilon^{\lambda\mu\nu}$ is the totally antisymmetric tensor density, satisfying $\epsilon^{123} = 1$ and $\epsilon_{123} = g := \det(g_{\mu\nu})$. It is worthwhile to note that $\mathcal{L}_{\text{top}}$ is manifestly metric-independent as the Christoffel connection is torsion-free, and that $D_\mu X^\mu$ is effectively the only term in $L_g$ which contains the Christoffel connection after replacing the fermionic term $\bar{\eta}D^{-}_{\mu}\chi^\mu$ by $-\bar{\chi}^\mu D^{-}_{\mu}\eta$. The introduction of the curved background metric is necessary for the twisted action to lead to a 'topological' theory, in the sense of the metric independence.

Moreover, we have complexified the gauge field:

$$\tilde{A}^{+}_{\mu ab} := \tilde{A}_{\mu ab} - i\frac{1}{2\sqrt{g}}\,\epsilon_{\mu\nu\lambda}X^\nu_c X^\lambda_d f^{cd}{}_{ab}\,, \qquad \tilde{A}^{-}_{\mu ab} := \tilde{A}_{\mu ab} + i\frac{1}{2\sqrt{g}}\,\epsilon_{\mu\nu\lambda}X^\nu_c X^\lambda_d f^{cd}{}_{ab}\,, \tag{2.24}$$

such that

$$D^{+}_{\mu} = D_\mu + i\frac{1}{2\sqrt{g}}\,\epsilon_{\mu\nu\lambda}\left[X^\nu, X^\lambda, \quad\right], \qquad D^{-}_{\mu} = D_\mu - i\frac{1}{2\sqrt{g}}\,\epsilon_{\mu\nu\lambda}\left[X^\nu, X^\lambda, \quad\right], \tag{2.25}$$

and

$$\begin{aligned}\tilde{F}^{+}_{\mu\nu}{}^{a}{}_{b} &= \tilde{F}_{\mu\nu}{}^{a}{}_{b} - i\frac{1}{\sqrt{g}}\epsilon_{\nu\rho\sigma}(D_\mu X^\rho)_c X^\sigma_d f^{cda}{}_b + i\frac{1}{\sqrt{g}}\epsilon_{\mu\rho\sigma}(D^{+}_{\nu}X^\rho)_c X^\sigma_d f^{cda}{}_b\\ &= \tilde{F}_{\mu\nu}{}^{a}{}_{b} - i\frac{1}{\sqrt{g}}\epsilon_{\nu\rho\sigma}(D^{+}_{\mu}X^\rho)_c X^\sigma_d f^{cda}{}_b + i\frac{1}{\sqrt{g}}\epsilon_{\mu\rho\sigma}(D_\nu X^\rho)_c X^\sigma_d f^{cda}{}_b\,.\end{aligned} \tag{2.26}$$

It is worth while to note:

$$
\begin{aligned}
D_\lambda X_\mu - D_\mu X_\lambda &= D^+_\lambda X_\mu - D^+_\mu X_\lambda = D^-_\lambda X_\mu - D^-_\mu X_\lambda \,, \\
D^\mu X_\mu + i\frac{1}{6\sqrt{g}}\epsilon^{\mu\nu\rho}[X_\mu, X_\nu, X_\rho] &= D^+_\mu X^\mu - i\frac{1}{3\sqrt{g}}\epsilon_{\mu\nu\lambda}[X^\mu, X^\nu, X^\lambda] \,.
\end{aligned} \tag{2.27}
$$

The Euler-Lagrange equations of motion are, for bosons $X_\mu, Y^i, A^+_\mu$:

$$
\begin{aligned}
&D_\mu\left(D^\lambda X^\mu - D^\mu X^\lambda\right) + D^{-\lambda}\left(D^\mu X_\mu + i\frac{1}{6\sqrt{g}}\epsilon^{\mu\nu\rho}[X_\mu, X_\nu, X_\rho]\right) - i[\bar\eta, \gamma^i\chi^\lambda, Y_i] \\
&+i\frac{1}{\sqrt{g}}\epsilon^{\lambda\mu\nu}\left(2[\bar\eta, X_\mu, \chi_\nu] - [D^+_\mu Y^i, Y_i, X_\nu] + \frac{1}{2}[\bar\chi_\mu, \gamma^i\chi_\nu, Y_i]\right) - \frac{1}{2}[Y^i, Y^j, [X^\lambda, Y_i, Y_j]] = 0\,, \\
&D_\mu D^{+\mu}Y^i - i\frac{1}{\sqrt{g}}\epsilon^{\lambda\mu\nu}\left([D_\mu X_\nu, X_\lambda, Y_i] - \frac{1}{2}[\bar\chi_\mu, \gamma^i\chi_\nu, X_\lambda]\right) + i[\bar\eta, X_\mu, \gamma^i\chi^\mu] \\
&-i\frac{1}{2}[\bar\eta, \gamma^{ij}\eta, Y_j] - i\frac{1}{2}[\bar\chi^\mu, \gamma^{ij}\chi_\mu, Y_j] + \frac{1}{2}\Big[Y_j, Y_k, [Y^j, Y^k, Y^i]\Big] + \big[X_\mu, Y_j, [X^\mu, Y^j, Y^i]\big] = 0\,, \\
&f^{abcd}\Big(\frac{1}{2\sqrt{g}}\epsilon^{\mu\nu\lambda}\bar\chi_{\nu c}\chi_{\lambda d} - Y^i_c D^+_\mu Y_{id} + i\frac{1}{2\sqrt{g}}Y^i_c\epsilon^{\mu\nu\lambda}[\chi_\nu, X_\lambda, Y_i]_d - X_{\nu c}(D^\mu X^\nu - D^\nu X^\mu)_d \\
&\qquad + (D_\lambda X^\lambda + i\frac{1}{6\sqrt{g}}\epsilon^{\lambda\mu\nu}[X_\mu, X_\nu, X_\lambda])_c X^\mu_d + \bar\eta_c\chi^\mu_d\Big) + i\frac{1}{\sqrt{g}}\epsilon^{\mu\nu\lambda}\tilde{F}^{+ab}_{\nu\lambda} = 0\,,
\end{aligned} \tag{2.28}
$$

and for fermions $\eta, \chi_\mu$ :

$$
\begin{aligned}
\zeta &:= D^-_\mu\chi^\mu + i[Y^i, X^\mu, \gamma_i\chi_\mu] + i\frac{1}{2}[Y^i, Y^j, \gamma_{ij}\eta] = 0\,, \\
\xi_\mu &:= D^-_\mu\eta + i[Y^i, X_\mu, \gamma_i\eta] - i\frac{1}{2}[Y^i, Y^j, \gamma_{ij}\chi_\mu] \\
&\quad + \frac{1}{\sqrt{g}}\epsilon_\mu{}^{\nu\lambda}D^+_\nu\chi_\lambda - i\frac{1}{\sqrt{g}}\epsilon_{\mu\nu\lambda}[Y^i, X^\nu, \gamma_i\chi^\lambda] = 0\,.
\end{aligned} \tag{2.29}
$$

### 2.4 On-shell scalar supersymmetry and BPS equations

In flat background, the twisted Bagger-Lambert-Gustavsson action (2.21) is invariant under the sixteen supersymmetries $Q^\alpha, Q^\alpha_\mu$ as in the untwisted case. However, in curved backgrounds, in order to have supersymmetry unbroken, it is necessary that the corresponding supersymmetry parameters $\varepsilon^\alpha$, $\varepsilon^\alpha_\mu$ should be covariantly constant. Generically, this requirement can be only met for the scalar parameters $\varepsilon^\alpha$. Indeed, for our twisted Bagger-Lambert-Gustavsson theory in a generic curved background, the twelve vectorial supersymmetries are broken and only the four scalar supersymmetries survive. Explicitly the unbroken scalar supersymmetries are given by:

$$
\begin{aligned}
\delta X_\mu &= \bar\varepsilon\chi_\mu\,, \\
\delta Y^i &= \bar\varepsilon\gamma^i\eta\,, \\
\delta\eta &= -\left(D_\mu X^\mu + i\frac{1}{6\sqrt{g}}\epsilon_{\mu\nu\lambda}[X^\mu, X^\nu, X^\lambda]\right)\varepsilon + i\frac{1}{6}[Y^i, Y^j, Y^k]\gamma_{ijk}\varepsilon\,,
\end{aligned} \tag{2.30}
$$

$$
\begin{aligned}
\delta\chi_\lambda &= \frac{1}{\sqrt{g}}\epsilon_{\lambda\mu\nu}D^\mu X^\nu\varepsilon + D^+_\lambda Y^i\gamma_i\varepsilon + i\frac{1}{2}[Y^i,Y^j,X_\lambda]\gamma_{ij}\varepsilon\,,\\
\delta\tilde{A}_{\mu ab} &= i\left(-X_{\mu c}\bar{\varepsilon}\eta_d + \frac{1}{\sqrt{g}}\epsilon_{\mu\nu\lambda}X^\nu_c\bar{\varepsilon}\chi^\lambda_d + Y_{ic}\bar{\varepsilon}\gamma^i\chi_{\mu d}\right)f^{cd}{}_{ab}\,.
\end{aligned}
$$

Equivalently in terms of scalar supercharges:

$$
\begin{aligned}
[Q^\alpha,X_\mu] &= \chi^\alpha_\mu\,,\\
\left[Q^\alpha,Y^i\right] &= (\gamma^i\eta)^\alpha\,,\\
\{Q^\alpha,\bar{\eta}_\beta\} &= -\left(D_\mu X^\mu + i\frac{1}{6\sqrt{g}}\epsilon_{\mu\nu\lambda}[X^\mu,X^\nu,X^\lambda]\right)\delta^\alpha{}_\beta - i\frac{1}{6}[Y^i,Y^j,Y^k](\gamma_{ijk})^\alpha{}_\beta\,, \\
\{Q^\alpha,\bar{\chi}_{\lambda\beta}\} &= \frac{1}{\sqrt{g}}\epsilon_\lambda{}^{\mu\nu}D_\mu X_\nu\delta^\alpha{}_\beta + D^+_\lambda Y^i(\gamma_i)^\alpha{}_\beta - i\frac{1}{2}[Y^i,Y^j,X_\lambda](\gamma_{ij})^\alpha{}_\beta\,,\\
\left[Q^\alpha,\tilde{A}_{\mu ab}\right] &= i\left(-X_{\mu c}\eta^\alpha_d + \frac{1}{\sqrt{g}}\epsilon_\mu{}^{\nu\lambda}X_{\nu c}\chi^\alpha_{\lambda d} + Y_{ic}(\gamma^i\chi_{\mu d})^\alpha\right)f^{cd}{}_{ab}\,.
\end{aligned}
\qquad (2.31)
$$

Successive scalar supersymmetry transformations give

$$
\begin{aligned}
\left[\{Q^\alpha,Q^\beta\},X_\mu\right] &= i[Y^i,Y^j,X_\mu](\gamma_{ij}C^{-1})^{\alpha\beta}\,,\\
\left[\{Q^\alpha,Q^\beta\},Y^i\right] &= i[Y^i,Y^j,Y^i](\gamma_{ij}C^{-1})^{\alpha\beta}\,,\\
\left[\{Q^\alpha,Q^\beta\},\eta_\gamma\right] &= i[Y^i,Y^j,\eta_\gamma](\gamma_{ij}C^{-1})^{\alpha\beta} + \delta^\beta{}_\gamma\zeta^\alpha + \delta^\alpha{}_\gamma\zeta^\beta\,, \\
\left[\{Q^\alpha,Q^\beta\},\chi_{\mu\gamma}\right] &= i[Y^i,Y^j,\chi_{\mu\gamma}](\gamma_{ij}C^{-1})^{\alpha\beta} - \delta^\beta{}_\gamma\xi^\alpha_\mu - \delta^\alpha{}_\gamma\xi^\beta_\mu\,,\\
\left[\{Q^\alpha,Q^\beta\},\tilde{A}_{\mu ab}\right] &= 2i\left(Y^i_c D_\mu Y^j_d(\gamma_{ij}C^{-1})^{\alpha\beta}\right)f^{cd}{}_{ab}\,.
\end{aligned}
\qquad (2.32)
$$

Apart from the Euler-Lagrange equations of the fermions, $\zeta,\xi_\mu$ (2.29), the right hand sides in (2.33) correspond precisely to the gauge transformation (2.5). Thus, *the scalar supercharges are nilpotent on-shell up to gauge transformations.*

From the supersymmetry transformations of the fermions (2.31), we see that supersymmetric invariant bosonic configurations must satisfy the following BPS conditions:[3]

$$
\begin{aligned}
&D^+_\mu X^\mu - i\frac{1}{3\sqrt{g}}\epsilon_{\mu\nu\lambda}[X^\mu,X^\nu,X^\lambda] = 0\,, \qquad D^+_\mu X_\nu - D^+_\nu X_\mu = 0\,,\\
&D^+_\mu Y^i = 0\,, \qquad [Y^i,Y^j,Y^k] = 0\,, \qquad [Y^i,Y^j,X_\lambda] = 0\,.
\end{aligned}
\qquad (2.33)
$$

Further, these BPS conditions imply the bosonic Euler-Lagrange equations of motion (2.28) if and only if

$$
F^+_{\mu\nu} = 0\,. \qquad (2.34)
$$

## 3. Off-shell supersymmetric formulation of the twist

The above on-shell formulation of the twist is not yet sufficient to define a genuine topological field theory which depends only on the topology of the three-dimensional base

[3]For various BPS states in the original untwisted BLG theory, including the classification, we refer [22–24].

manifold, since the scalar supercharges are only on-shell nilpotent and the scalar supersymmetry transformations (2.31) are not independent from the base manifold metric. In this section we construct an off-shell supersymmetric formalism of the twist which will eventually lead to a genuine topological field theory.

### 3.1 Off-shell supersymmetry algebra

Our off-shell supersymmetric formulation requires two auxiliary fields which we call $h$ and $h_\mu$. The off-shell $Q$-variations are defined over $\{X_\mu, Y^i, h, h_\mu, \eta, \chi_\mu, A^+_\mu\}$ as follows:[4]

$$\begin{aligned}
[Q^\alpha, X_\mu] &= \chi^\alpha_\mu\,,\\
\left[Q^\alpha, Y^i\right] &= (\gamma^i\eta)^\alpha\,,\\
[Q^\alpha, h] &= -i\frac{1}{2}[Y^i, Y^j, (\gamma_{ij}\eta)^\alpha]\,,\\
[Q^\alpha, h_\mu] &= -D^+_\mu\eta^\alpha + i[X_\mu, Y^i, (\gamma_i\eta)^\alpha] + i\frac{1}{2}[Y^i, Y^j, (\gamma_{ij}\chi_\mu)^\alpha]\,,\\
\{Q^\alpha, \bar\eta_\beta\} &= -h\delta^\alpha{}_\beta - i\frac{1}{6}[Y^i, Y^j, Y^k](\gamma_{ijk})^\alpha{}_\beta\,,\\
\{Q^\alpha, \bar\chi_{\mu\beta}\} &= h_\mu\delta^\alpha{}_\beta + D^+_\mu Y^i(\gamma_i)^\alpha{}_\beta - i\frac{1}{2}[Y^i, Y^j, X_\mu](\gamma_{ij})^\alpha{}_\beta\,,\\
\left[Q^\alpha, \tilde A^+_{\mu ab}\right] &= i\left(-X_{\mu c}\eta^\alpha_d + Y_{ic}(\gamma^i\chi_{\mu d})^\alpha\right) f^{cd}{}_{ab}\,.
\end{aligned} \tag{3.1}$$

It is straightforward to verify that our $Q$-variations (3.1) are nilpotent strictly off-shell, up to a gauge transformation: for all the fields in $\{X_\mu, Y^i, h, h_\mu, \eta, \chi_\mu, A^+_\mu\}$, we find

$$Q^2 = \text{gauge transformation}\,, \tag{3.2}$$

where, with an arbitrary constant $c$-number spinor $v_\alpha$, $Q = \bar v_\alpha Q^\alpha$ and the gauge parameter (2.5) is given by

$$\Lambda_{ab} = i\frac{1}{2}Y^i_a Y^j_b(\gamma_{ij}C^{-1})^{\alpha\beta}\bar v_\alpha\bar v_\beta\,. \tag{3.3}$$

Here the off-shell supersymmetry algebra is defined for $\tilde A^+_{\mu ab}$ and not for $\tilde A^-_{\mu ab}$. In our off-shell supersymmetric formalism it is not necessary to define the $Q^\alpha$-variation of $\tilde A^-_{\mu ab} = (\tilde A^+_{\mu ab})^*$. In fact, in our off-shell supersymmetric formulation we may relax the decomposition rule of the complex gauge field into the real and imaginary parts given in eq. (2.24), such that $\tilde A_{\mu ab}$ will never appear and we may keep only the reality condition $\tilde A^-_{\mu ab} = (\tilde A^+_{\mu ab})^*$.[5] In this case, the identities (2.27) do not hold anymore.

[4] Transforming (2.31) to (3.1), we made the identification,

$$h \equiv D^{+\mu}X_\mu - i\frac{1}{3\sqrt{g}}\epsilon^{\mu\nu\lambda}[X_\mu, X_\nu, X_\lambda]\,, \qquad h_\mu \equiv \frac{1}{\sqrt{g}}\epsilon_{\mu\nu\lambda}D^{+\nu}X^\lambda\,,$$

where "$\equiv$" means on-shell equivalence. To obtain the $Q$-variation of the auxiliary fields, we take the variation of their on-shell values and use the equations of motion to remove any metric-dependent terms.

[5] Note that $[Q^\alpha, \tilde A^+_{\mu ab}]^\dagger$ does not lead to the $Q^\alpha$-variation of $\tilde A^-_{\mu ab} = (\tilde A^+_{\mu ab})^*$, since $Q^\alpha$ is not hermitian.



JHEP11(2008)014

**Ghost number.** In topological field theories, it is often useful to introduce the so-called ghost-number $U$, though it may *not* lead to a symmetry of the topological action, as will be the case with our twisted Lagrangian. We first assign ghost number one to the scalar supercharges, $U(Q)=1$. Then (3.1) uniquely determines the ghost number of each field:

$$U(X_\mu,\,\chi_\mu,\,h_\mu,\,Y,\,\eta,\,h,\,\tilde{A}^+_\mu)=(-1,\,0,\,+1,\,+1,\,+2,\,+3,\,0)\,. \tag{3.4}$$

### 3.2 Off-shell supersymmetric Lagrangian

Provided the off-shell supersymmetry algebra, it is straightforward to obtain the off-shell supersymmetric Lagrangian:

$$\begin{aligned}\mathcal{L}_{\text{off-shell}}=&i\epsilon^{\mu\nu\lambda}\left(\frac{1}{2}A^+_{\mu ab}\partial_\nu\tilde{A}^{+ab}_\lambda+\frac{1}{3}A^+_{\mu ab}\tilde{A}^{+a}_{\nu\ \ c}\tilde{A}^{+cb}_\lambda\right)\\
&-\epsilon^{\mu\nu\lambda}\mathrm{Tr}\left(\frac{1}{2}\bar{\chi}_\mu D^+_\nu\chi_\lambda-i\frac{1}{2}\bar{\chi}_\mu\left[\gamma_i\chi_\nu,X_\lambda,Y^i\right]+i\frac{1}{3}h[X_\mu,X_\nu,X_\lambda]-h_\mu D^+_\nu X_\lambda\right)\\
&+\sqrt{g}\,\mathrm{Tr}\left(\frac{1}{2}D^+_\mu Y^iD^-_\mu Y_i-\frac{1}{2}h^2+hD^{+\mu}X_\mu-\frac{1}{2}h^\mu h_\mu\right.\\
&\qquad+\frac{1}{12}[Y^i,Y^j,Y^k][Y_i,Y_j,Y_k]+\frac{1}{4}[X_\mu,Y^j,Y^k][X^\mu,Y^j,Y^k]\\
&\qquad-\bar{\chi}^\mu D^-_\mu\eta+i\bar{\eta}\gamma_i\left[Y^i,X_\mu,\chi^\mu\right]+i\frac{1}{4}\bar{\eta}\gamma_{ij}\left[Y^i,Y^j,\eta\right]\\
&\qquad\left.+i\frac{1}{4}\bar{\chi}^\mu\gamma_{ij}\left[Y^i,Y^j,\chi_\mu\right]\right).\end{aligned} \tag{3.5}$$

Integrating out the auxiliary fields, the above off-shell supersymmetric Lagrangian (3.5) reduces to the form:

$$\begin{aligned}\mathcal{L}_{\text{off-shell}}\equiv\ &i\epsilon^{\mu\nu\lambda}\left(\frac{1}{2}A^+_{\mu ab}\partial_\nu\tilde{A}^{+ab}_\lambda+\frac{1}{3}A^+_{\mu ab}\tilde{A}^{+a}_{\nu\ \ c}\tilde{A}^{+cb}_\lambda\right)\\
&-\epsilon^{\mu\nu\lambda}\mathrm{Tr}\left(\frac{1}{2}\bar{\chi}_\mu D^+_\nu\chi_\lambda-i\frac{1}{2}\bar{\chi}_\mu\left[\gamma_i\chi_\nu,X_\lambda,Y^i\right]\right)\\
&+\sqrt{g}\,\mathrm{Tr}\left(\frac{1}{2}(D^{+\mu}X_\mu-i\frac{1}{3\sqrt{g}}\epsilon^{\mu\nu\lambda}[X_\mu,X_\nu,X_\lambda])^2\right.\\
&\qquad+\frac{1}{4}(D^+_\mu X_\nu-D^+_\nu X_\mu)(D^{+\mu}X^\nu-D^{+\nu}X^\mu)\\
&\qquad+\frac{1}{2}D^+_\mu Y^iD^-_\mu Y_i+\frac{1}{12}[Y^i,Y^j,Y^k][Y_i,Y_j,Y_k]\\
&\qquad+\frac{1}{4}[X_\mu,Y^j,Y^k][X^\mu,Y^j,Y^k]-\bar{\chi}^\mu D^-_\mu\eta+i\bar{\eta}\gamma_i\left[Y^i,X_\mu,\chi^\mu\right]\\
&\qquad\left.+i\frac{1}{4}\bar{\eta}\gamma_{ij}\left[Y^i,Y^j,\eta\right]+i\frac{1}{4}\bar{\chi}^\mu\gamma_{ij}\left[Y^i,Y^j,\chi_\mu\right]\right),\end{aligned}$$

which is very similar, but not identical, to the on-shell supersymmetric Lagrangian (2.22), (2.23). Only if we assume the decomposition of the complex gauge field into the real and imaginary parts given in (2.24), they coincide.

A crucial feature of the off-shell supersymmetric Lagrangian (3.5) is that it can be written as a sum of $Q$-closed and $Q$-exact parts:

$$\mathcal{L}_{\text{off-shell}}=\mathcal{L}_{\text{closed}}+\{Q,\Sigma\}\,, \tag{3.6}$$

where, firstly with a pair of arbitrary constant $c$-number spinors $\bar{v}_\alpha, u^\beta$ satisfying $\bar{v}_\alpha u^\alpha \neq 0$, the scalar supercharge and and the fermionic scalar in the $Q$-exact part are

$$Q = \bar{v}_\alpha Q^\alpha \,, \qquad \Sigma = \bar{\Sigma}_\alpha u^\alpha / (\bar{v}_\beta u^\beta) \,, \tag{3.7}$$

of which the fermionic SO(5) spinor is given by

$$\begin{aligned} \bar{\Sigma} = & \frac{1}{2} h \bar{\eta} - \frac{1}{2} h^\mu \bar{\chi}_\mu + \frac{1}{2} (D^+_\mu Y^i) \bar{\chi}^\mu \gamma_i - (D^+_\mu X^\mu) \bar{\eta} \\ & - i \frac{1}{4} [Y^i, Y^j, X_\mu] \bar{\chi}^\mu \gamma_{ij} - i \frac{1}{12} [Y^i, Y^j, Y^k] \bar{\eta} \gamma_{ijk} \,. \end{aligned} \tag{3.8}$$

The $Q$-closed part is then

$$\begin{aligned} \mathcal{L}_{\text{closed}} = & \, i \epsilon^{\mu\nu\lambda} \left( \frac{1}{2} f^{abcd} A^+_{\mu ab} \partial_\nu A^+_{\lambda cd} - \frac{1}{3} f^{cdag} f^{efb}{}_g A^+_{\mu ab} A^+_{\nu cd} A^+_{\lambda ef} \right) \\ & + \epsilon^{\mu\nu\lambda} \text{Tr} \bigg( - \frac{1}{2} \bar{\chi}_\mu D^+_\nu \chi_\lambda + h_\mu D^+_\nu X_\lambda + i \frac{1}{2} \bar{\chi}_\mu [\gamma_i \chi_\nu, X_\lambda, Y^i] \\ & \qquad - i \frac{1}{3} h [X_\mu, X_\nu, X_\lambda] - i \bar{\eta} [\chi_\mu, X_\nu, X_\lambda] - \frac{i}{2} D^+_\mu Y^i [X_\nu, X_\lambda, Y_i] \bigg) \,. \end{aligned} \tag{3.9}$$

Direct manipulation indeed shows that $\mathcal{L}_{\text{closed}}$ is $Q$-closed up to total derivative terms, and more interestingly about the $Q$-exact term,

$$\begin{aligned} \{Q^\alpha, \bar{\Sigma}_\beta\} = & \, \delta^\alpha{}_\beta \{Q, \Sigma\} \\ = & \, \delta^\alpha{}_\beta \text{Tr} \bigg[ \frac{1}{2} D^+_\mu Y^i D^{+\mu} Y_i - \frac{1}{2} h^2 + h D^{+\mu} X_\mu - \frac{1}{2} h^\mu h_\mu - \bar{\chi}^\mu D^-_\mu \eta + i \bar{\eta} \gamma^i [Y^i, X_\mu, \chi^\mu] \\ & + i \frac{1}{4} \bar{\eta} \gamma_{ij} [Y^i, Y^j, \eta] + i \frac{1}{\sqrt{g}} \epsilon^{\mu\nu\lambda} \bar{\eta} [\chi_\mu, X_\nu, X_\lambda] + i \frac{1}{4} \bar{\chi}^\mu \gamma_{ij} [Y^i, Y^j, \chi_\mu] \\ & + \frac{1}{12} [Y^i, Y^j, Y^k][Y_i, Y_j, Y_k] + \frac{1}{4} [Y^i, Y^j, X^\mu][Y_i, Y_j, X_\mu] \bigg] \,. \end{aligned} \tag{3.10}$$

In fact, utilizing the existing SO(5) symmetry of the action, one can rotate the constant $c$-number spinor such that only one component is nontrivial e.g. $\bar{v}_\alpha = v \delta_{\alpha 1}$. In this case, from (3.10) only the corresponding one component of $\bar{\Sigma}_\alpha$, i.e. $\bar{\Sigma}_1$ couples to the supercharge and contributes to the formation of the $Q$-exact part of the Lagrangian. In this way, different choices of the linear combination of the four scalar supercharges (3.7) are all SO(5) equivalent. At this point it is worthwhile to compare with a topological twisting of $\mathcal{N} = 4$ super Yang-Mills theory [16, 17, 21] where there appears a pair of scalar supercharges. In contrast to our case, the twisted action possesses no $R$-symmetry which would rotate the two supercharges to each other. Hence, a different linear combination of the scalar supercharges defines inequivalent cohomology.

The $Q$-closed part has the ghost number zero and contains no metric dependent term, being explicitly topological, c.f. (2.22). On the other hand, the $Q$-exact part has no definite ghost number and contains explicitly metric dependent terms. In fact, all the metric dependence of the off-shell supersymmetric Lagrangian (3.5) can be read off from $\bar{\Sigma}$, because the $Q$-transformations (3.1) are independent of the base manifold metric. Thus, the energy-momentum tensor is $Q$-exact, and our off-shell supersymmetric formulation of the Bagger-Lambert-Gustavsson action indeed defines a genuine topological field theory in three dimensions. Note that since $\bar{\Sigma}$ does not involve $A_\mu^-$, our $Q$-transformation rule - which is defined over $\{X_\mu, Y^i, h, h_\mu, \eta, \chi_\mu, A_\mu^+\}$ only - can be applied to it.

Fierz identities have been heavily used for the derivation of the above formulae. We summarize them in the appendix.

## 4. Observables and partition function

### 4.1 Observables

As is well-known, a local operator that is $Q$-closed up to total derivatives leads to a series of observables. For instance, we can have a relation:

$$[Q, \mathcal{O}_3] = \mathrm{d}\mathcal{O}_2\,, \qquad \{Q, \mathcal{O}_2\} = \mathrm{d}\mathcal{O}_1\,, \qquad [Q, \mathcal{O}_1] = \mathrm{d}\mathcal{O}_0\,, \qquad \{Q, \mathcal{O}_0\} = 0\,. \tag{4.1}$$

Here, $\mathcal{O}_n$ are $n$-forms with alternating statistics. The first relation holds by assumption, and the rest follows from the nilpotency of $Q$. The integration of $\mathcal{O}_n$ over an $n$-cycle then gives a well-defined observable.

One particular family of observables that comes free for any topological theory is the one associated with the $Q$-closed part of the off-shell supersymmetric Lagrangian. For our theory, we find

$$[Q^\alpha, \mathcal{L}_{\mathrm{closed}}] = \partial_\mu \mathcal{L}^{\alpha\mu}\,, \qquad \left[Q^{(\alpha}, \mathcal{L}^{\beta)\mu}\right] = \partial_\nu \mathcal{L}^{\alpha\beta\mu\nu}\,, \qquad \left[Q^{(\alpha}, \mathcal{L}^{\beta\gamma)}_{\mu\nu}\right] = 0\,, \tag{4.2}$$

where the brackets denote the symmetrization of the spinorial indices with weight one, i.e. $A^{(\alpha}B^{\beta)} = \frac{1}{2}(A^\alpha B^\beta + A^\beta B^\alpha)$ and

$$\begin{aligned}\mathcal{L}^\alpha_\mu &= -\epsilon_{\mu\nu\lambda}\mathrm{Tr}\left(i\frac{1}{4}[Y^i, Y^j, (\gamma_{ij}\chi^\nu)^\alpha]X^\lambda + i\frac{1}{2}[X^\nu, Y^i, (\gamma_i\eta)^\alpha]X^\lambda + \frac{1}{2}\eta^\alpha D^{+\nu}X^\lambda - \frac{1}{2}\chi^{\alpha\nu}h^\lambda\right),\\ \mathcal{L}^{\alpha\beta\mu\nu} &= \epsilon^{\mu\nu\lambda}\mathrm{Tr}\left(\frac{1}{2}\eta^{(\alpha}\chi^{\beta)}_\lambda + i\frac{1}{12}[Y^i, Y^j, Y^k]X_\lambda(\gamma_{ijk}C)^{\alpha\beta}\right).\end{aligned} \tag{4.3}$$

**Wilson loop.** Wilson loop operator is one of the most fundamental observables in any non-Abelian gauge theory. Moreover, the Wilson loop in pure Chern-Simons theory [18] has been used to compute knot invariants of three manifolds. So, it is natural to ask whether we can have a sort of Wilson loop as an observable in our case too.

The simplest Wilson loop made of $\tilde{A}^+$ is not a good observable since $\tilde{A}^+$ is not $Q$-closed,

$$\left[Q^\alpha, \tilde{A}^+_{\mu ab}\right] = i\left(-X_{\mu c}\eta^\alpha_d + Y_{ic}(\gamma^i\chi_{\mu d})^\alpha\right) f^{cd}{}_{ab}\,.$$

It is tempting to modify $\tilde{A}^+$ further to make it $Q$-closed, but it appears that it is not possible to do so and there is no Wilson loop-like observable in the twisted Bagger-Lambert-Gustavsson theory. The argument goes as follows. Consider $\tilde{A}^+_{\mu ab} \;\rightarrow\; \mathcal{A}_{\mu ab} = \tilde{A}^+_{\mu ab} + B_{\mu ab}$ with $B_{\mu ab} \equiv i f^{cd}{}_{ab}X_{\mu c}Y_{id}s^i$, where $s^i$ is a constant vector of SO(5). Requiring closedness of $\mathcal{A}_{\mu ab}$ under $v_\alpha Q^\alpha$, we find two conditions:

$$\text{(a)} \quad s^i v_\alpha = v_\beta(\gamma^i)^\beta{}_\alpha\,, \qquad \text{(b)} \quad v_\alpha = s^i v_\beta(\gamma^i)^\beta{}_\alpha\,.$$

Condition (b) is not very strong; for a given $v_\alpha$, it is easy to choose an $s^i$ satisfying it. On the other hand, condition (a) is very strong. Using the SO(5) covariance, we can always go to a basis in which, say, $s^1 = 0$. Then, we have $0 = v_\beta(\gamma^1)^\beta{}_\alpha$, which implies an unacceptable condition $v_\alpha = 0$ because $\gamma^1$ is invertible.

### 4.2 Partition function

Let us now consider the partition function in a quantum field theory in general:

$$\mathcal{Z} = \int \mathcal{D}\Phi \ \exp(-\mathcal{S})\,. \tag{4.4}$$

In the usual semi-classical expansion, one proceeds in four steps: classical action, one loop determinants, higher loop corrections and non-perturbative instanton corrections. However, in topological quantum field theory, one loop correction alone around all the BPS configurations can lead to an exact result due to the localization.

The localization follows from the fact that the partition function (4.4) and the vev of observables are invariant under the smooth deformation of the Lagrangian,

$$\mathcal{L} = \mathcal{L}_{\text{closed}} + \{Q,\Sigma\} \quad \rightarrow \quad \mathcal{L}_t = \mathcal{L}_{\text{closed}} + t\{Q,\Sigma\}\,, \tag{4.5}$$

with an arbitrary real parameter $t$. For large positive values of $t$, the path integral is "localized" to field configurations with $\{Q,\Sigma\} = 0$. The expression for $\{Q,\Sigma\}$ (3.10) implies that the path integral localizes to

$$D^+_\mu X^\mu = 0\,, \quad h_\mu = 0\,, \quad D^+_\mu Y^i = 0\,, \quad [Y^i,Y^j,Y^k] = 0\,, \quad [Y^i,Y^j,X_\lambda] = 0\,. \tag{4.6}$$

A systematic study of the full partition function, including the integral over the BPS configurations, is out of the scope of this work. Even at the classical level, we would have to deal with the subtleties due to the imaginary value and gauge non-invariance of the Chern-Simons term; see [25] for a recent discussion. Here, as a first step forward, we evaluate the one loop determinants in the trivial background with vanishing vev for all fields.

When the three-algebra is equipped with a positive definite norm, the possible dimension of the three-algebra is either one (trivial case), four or infinity. Either precisely for the trivial dimension, or effectively for the evaluation of the one-loop determinants in the nontrivial dimensions, we have copies of the following action for free fields (in the form notation),

$$\mathcal{S} = \int \mathrm{d}^3x\,\sqrt{g}\left[\frac{1}{2}\,(X_1,\Delta_1 X_1) + \frac{1}{2}(Y^i,\Delta_0 Y_i) - (\bar\chi_1,*\mathrm{d}\chi_1) - (\bar\eta,\mathrm{d}^\dagger\chi_1)\right], \tag{4.7}$$

where $\Delta_0$ and $\Delta_1$ are Laplacians acting on zero and one forms respectively,

$$\Delta_0 Y^i = -\nabla^\mu\nabla_\mu Y^i\,, \qquad \Delta_1 X_\mu = -\nabla^\nu\nabla_\nu X_\mu - [\nabla_\mu,\nabla^\nu]X_\nu\,. \tag{4.8}$$

Note also that $\sqrt{g}\,(\bar\chi_1,*\mathrm{d}\chi_1) = \bar\chi_1 \wedge \mathrm{d}\chi_1$. For the nontrivial three-algebra dimensions we omitted the Chern-Simons term of the gauge field, since at one loop level the contribution from the gauge field cancels out against those from the gauge-fixing ghosts.

In general, according to the Hodge theorem, any $p$-form, $\psi_p$, in a compact manifold of the positive definite signature decomposes uniquely into the harmonic form, $h_p$, exact form, $\mathrm{d}\alpha_{p-1}$, and coexact form, $\mathrm{d}^\dagger\beta_{p+1} = (-1)^{p+1} * \mathrm{d} * \beta_{p+1}$,

$$\psi_p = h_p + \mathrm{d}\alpha_{p-1} + \mathrm{d}^\dagger\beta_{p+1}\,, \tag{4.9}$$

where $h_p$, $\alpha_{p-1}$ and $\beta_{p+1}$ are all globally well defined. From the positive definiteness, we also have $\mathrm{d}h_p = 0$, $\mathrm{d}^\dagger h_p = 0$. The Laplacian on $p$-form i.e. $\Delta_p$ is given by[6]

$$\Delta_p = \mathrm{d}^\dagger\mathrm{d} + \mathrm{d}\mathrm{d}^\dagger\,, \tag{4.10}$$

so that each of $\mathrm{d}^\dagger\mathrm{d}$ and $\mathrm{d}\mathrm{d}^\dagger$ diagonalizes over the harmonic, exact and coexact $p$-form spaces.

In our case of the free action above (4.7), integrating out $\eta$ field forces to set $\mathrm{d}^\dagger\chi_1 = 0$, and hence with $\chi_1 = h_1 + \mathrm{d}\alpha_0 + \mathrm{d}^\dagger\beta_2$, from the positive definiteness the partition function saturates at

$$\mathrm{d}\alpha_0 = 0 \quad \text{for} \quad \chi_1\,, \tag{4.11}$$

which can be regarded as a gauge fixing in BRST quantization.

When there are fermionic zero modes, the bare partition function vanishes. In our case, there are one zero mode for $\eta$ and $b_1$ (the first Betti number) zero modes for $\chi_1$. Assuming that the right number of zero modes are absorbed by products of fermions from observables and/or interaction vertices, we find

$$\mathcal{Z}_{\mathrm{one-loop}} := \int \mathcal{D}X\mathcal{D}Y\mathcal{D}\chi\; e^{-\mathcal{S}} = \frac{\mathrm{Pf}\,[C(*\mathrm{d})_1]}{[\mathrm{det}\Delta_0]^{\frac{5}{2}}\,[\mathrm{det}\Delta_1]^{\frac{1}{2}}} = \frac{[\mathrm{det}\Delta_1]^{\frac{3}{2}}}{[\mathrm{det}\Delta_0]^{\frac{9}{2}}}\,, \tag{4.12}$$

where the second equality follows from [7] $\mathrm{Pf} = \sqrt{\mathrm{det}}$, $C^2 = -1_{4\times4}$, $\mathrm{det}(\mathrm{d}^\dagger\mathrm{d})_1 = \mathrm{det}\Delta_1/\mathrm{det}\Delta_0$. The final expression is nothing but the topological quantity known as the Ray-Singer torsion in three-dimensions:

$$\prod_{p=0}^{3}\,[\mathrm{det}\Delta_p]^{-(-1)^p\frac{1}{2}p} = \frac{[\mathrm{det}\Delta_0]^{\frac{3}{2}}}{[\mathrm{det}\Delta_1]^{\frac{1}{2}}} = \mathcal{Z}_{\mathrm{one-loop}}^{-\frac{1}{3}}\,. \tag{4.13}$$

We close this section with a comparison with a similar computation in the Rozansky-Witten theory [19]. The fermionic part of our free action (4.7) is essentially identical to that of Rozansky-Witten theory. The bosonic part of Rozansky-Witten theory is a non-linear sigma model with a hyper-Kähler target space, so it is quite different from our theory. Nevertheless, the combination of the bosonic and fermionic contributions of Rozansky-Witten theory also gives rise to the Ray-Singer torsion but with a different power from ours, i.e. $-1/2$ versus $-1/3$.

---

[6] Explicitly we have for a $p$-form, $\psi$,

$$(\mathrm{d}\psi)_{a_1a_2\cdots a_{p+1}} = (p+1)\nabla_{[a_1}\psi_{a_2a_3\cdots a_{p+1}]}\,, \quad (\mathrm{d}^\dagger\psi)_{a_1a_2\cdots a_{p-1}} = -\nabla^b\psi_{b\,a_1a_2\cdots a_{p-1}}\,,$$
$$(\Delta_p\psi)_{a_1a_2\cdots a_p} = -\nabla^b\nabla_b\psi_{a_1a_2\cdots a_p} + p[\nabla_b,\nabla_{[a_1}]\psi^b{}_{a_2a_3\cdots a_p]}\,.$$

[7] In general for a $p$-form in $d$ dimension, we have

$$\mathrm{det}\Delta_p = \mathrm{det}\Delta_{d-p}\,, \quad \mathrm{det}\Delta_p = \mathrm{det}(\mathrm{d}^\dagger\mathrm{d})_p\mathrm{det}(\mathrm{d}\mathrm{d}^\dagger)_p\,, \quad \mathrm{det}(\mathrm{d}^\dagger\mathrm{d})_p = \mathrm{det}(\mathrm{d}\mathrm{d}^\dagger)_{p+1}\,.$$



JHEP11(2008)014

## 5. Relation to M5: partial topological twist of six-dimensional theory

If we introduce an auxiliary three manifold, an explicit realization of an infinite dimensional three-algebra follows straightforwardly from the Nambu three-bracket defined on the internal manifold. This suggests that Bagger-Lambert-Gustavsson theory with infinite dimensional gauge group describes M5-brane as a condensation of multiple M2-branes [12–14]. In fact, by generalizing the Brink-Di Vecchia-Howe-Polyakov method, Nambu-Goto action for a five-brane can be reformulated as a three-dimensional gauged nonlinear sigma model having a Nambu three-bracket squared potential [13].

Introducing a functional basis for the three-manifold $T^a(y)$, we let all the variables be functions on the whole six-dimensions e.g. $X_\mu(x,y) = X_{\mu a}(x)T^a(y)$. We represent the three-algebra by

$$\begin{aligned} [X,Y,Z] &\equiv \frac{1}{\sqrt{\hat{g}}}\epsilon^{\hat{\lambda}\hat{\mu}\hat{\nu}}\partial_{\hat{\lambda}}X\partial_{\hat{\mu}}Y\partial_{\hat{\nu}}Z\,, \\ D_\mu X &\equiv \partial_\mu X - A_{\mu ab}[T^a,T^b,X]\,, \\ \mathrm{Tr} &\equiv \int \mathrm{d}^3y\sqrt{\hat{g}}\,, \end{aligned} \tag{5.1}$$

where $\hat{g}$ is an arbitrary function of $x, y$ which can be identified as the determinant of the internal space metric $\hat{g}_{\hat{\mu}\hat{\nu}}(x,y)$. Then the whole six-dimensional space has the fiber bundle structure: at each point in $x$-space (base), there exists a corresponding internal $y$-space (fiber).

Now we recall the BPS equation:

$$D^+_\mu X^\mu = i\frac{1}{3\sqrt{g}}\epsilon_{\mu\nu\lambda}[X^\mu, X^\nu, X^\lambda]\,. \tag{5.2}$$

Provided the above Nambu-bracket realization of the three-algebra, this BPS equation reads

$$D^+_\mu X^\mu = i\frac{1}{3\sqrt{g}\sqrt{\hat{g}}}\epsilon_{\mu\nu\lambda}e^{\hat{\mu}\hat{\nu}\hat{\lambda}}\partial_{\hat{\mu}}X^\mu\partial_{\hat{\nu}}X^\nu\partial_{\hat{\lambda}}X^\lambda = i\frac{2}{\sqrt{\hat{g}}}\sqrt{\det(\partial_{\hat{\mu}}X^\lambda\partial_{\hat{\nu}}X_\lambda)}\,, \tag{5.3}$$

where the last equality holds since $\partial_{\hat{\mu}}X^\mu$ is a $3\times 3$ matrix. We integrate this formula over $y$-space or take the trace. The final expression then leads to the usual Gauss law in three-dimension:

$$\nabla\cdot E(x) = i\rho(x)\,, \tag{5.4}$$

with

$$\begin{aligned} E_\mu &= \frac{1}{2}\int \mathrm{d}^3y\,\sqrt{\hat{g}}\,X_\mu\,, \\ \rho &= \frac{1}{6\sqrt{g}}\epsilon_{\mu\nu\lambda}\int \mathrm{d}^3y\;\partial_{\hat{\mu}}\left(e^{\hat{\mu}\hat{\nu}\hat{\lambda}}X^\mu\partial_{\hat{\nu}}X^\nu\partial_{\hat{\lambda}}X^\lambda\right) = \int \mathrm{d}^3y\;\sqrt{\det(\partial_{\hat{\mu}}X^\lambda\partial_{\hat{\nu}}X_\lambda)}\,. \end{aligned} \tag{5.5}$$

Remarkably, the density $\rho(x)$ matches with the Nambu-Goto action having the $y$-space and the "$X^\mu$-space" as the world-volume and the target space. Reflecting upon the original untwisted BLG description of multiple M2-branes, $X^\mu$ corresponds to three transverse scalars and $x^\mu$ can be identified as three longitudinal physical directions in the static gauge. In our twisted theory with three-algebra realized by Nambu-bracket, it is then natural to regard the $(x,y)$-space and the $(x,X)$-space as the world-volume and the physical longitudinal space of an Euclidean M5-brane respectively with the partial static gauge "$x = x$". This M5-brane picture then reveals that any point-like BPS configuration in $x$-space or instanton may expand over $X$-space and in fact it corresponds to a Euclidean M2-brane. The $x$-space charge density $\rho(x)$ then measures the volume of a Euclidean M2-brane in the $X$-space. Furthermore, $\rho$ being a surface integral, if the $y$-space is compact, up to a $x$-space local factor, the density $\rho(x)$ counts the winding number of M2-branes wrapping three-cycles inside M5-brane. For a non-compact $y$-space, with a suitable boundary condition, the integral may not vanish too.

Since the $Q$-transformations involve the three-commutators and depend on the $y$-space metric, our twisted Bagger-Lambert-Gustavsson theory is topological only over the $x$-space but not over the $y$-space.

Furthermore with a six-dimensional metric:

$$\mathrm{d}s_6^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \hat{g}_{\hat{\mu}\hat{\nu}}\mathrm{d}x^{\hat{\mu}}\mathrm{d}x^{\hat{\nu}}\,, \tag{5.6}$$

if we define a "$(2,0)$" and a "$(0,2)$" two-form, respectively:

$$B_{\mu\nu} := \frac{1}{\sqrt{g}}\epsilon_{\mu\nu\lambda}X^\lambda\,, \quad B_{\hat{\mu}\hat{\nu}} := \frac{1}{3\sqrt{g}}\epsilon_{\mu\nu\lambda}\left(\partial_{\hat{\mu}}X^\mu\partial_{\hat{\nu}}X^\nu - \partial_{\hat{\nu}}X^\mu\partial_{\hat{\mu}}X^\nu\right)X^\lambda\,, \tag{5.7}$$

then in terms of their three-form field strengths,

$$\begin{aligned} H_{\lambda\mu\nu} &:= D^+_\lambda B_{\mu\nu} + D^+_\mu B_{\nu\lambda} + D^+_\nu B_{\lambda\mu} = \frac{1}{\sqrt{g}}\epsilon_{\lambda\mu\nu}D^+_\rho X^\rho\,,\\ H_{\hat{\lambda}\hat{\mu}\hat{\nu}} &:= \nabla_{\hat{\lambda}}B_{\hat{\mu}\hat{\nu}} + \nabla_{\hat{\mu}}B_{\hat{\nu}\hat{\lambda}} + \nabla_{\hat{\nu}}B_{\hat{\lambda}\hat{\mu}} = \frac{1}{3\sqrt{\hat{g}}\sqrt{g}}\epsilon_{\hat{\lambda}\hat{\mu}\hat{\nu}}\epsilon_{\lambda\mu\nu}[X^\lambda, X^\mu, X^\nu]\,, \end{aligned} \tag{5.8}$$

the BPS equation (5.2) can be written in a compact form:

$$H_{\lambda\mu\nu} = i\,(*H)_{\lambda\mu\nu}\,. \tag{5.9}$$

This corresponds to a partial self-duality equation of a three-form in Euclidean six dimension. It is partial, since it is the self-duality linking $(3,0)$ and $(0,3)$ field strength and the other one linking $(2,1)$ and $(1,2)$ is missing.[8]

Provided these dictionaries (despite of the incompleteness of the self-duality), the BPS equation (5.2) or (5.4) indeed realizes the coupling of the self-dual three-form to the M2-brane charge density.

[8]It seems hard to find the $(1,1)$ two-form $B_{\mu\hat{\mu}}$ which would complete the missing piece. One possible reason might be that the scalar supercharges are not SO(6) chiral in contrast to the supersymmetries of the six-dimensional M5-brane world-volume theory [26, 27].

## 6. Outlook

We have constructed a topological version of the BLG theory and took some preliminary steps to study its physical contents. But, clearly, more work would be required to reveal the full physical contents of the topological theory. First, an exhaustive list of observables should be found. Second, a systematic study of the BPS configurations and their contribution to the path integral should be done. Finally, a perturbative computation of the partition function and some of the observables should be carried out. We hope to address these issues in a future work.

Another obvious direction is to consider other related theories. In three dimensions, the minimum amount of supersymmetry needed to obtain a topological theory by twisting is $\mathcal{N} = 4$ i.e. eight supersymmetries. An SO(3) subgroup of the SO($\mathcal{N}$) $R$-symmetry should be combined with the "Lorentz" SO(3) to yield a nilpotent, scalar supercharge $Q$ of the twisted theory. But, for $\mathcal{N} = 3$, since the supercharge is a doublet of Lorentz SO(3) and a triplet of the $R$-symmetry SO(3), the twisting cannot give rise to a scalar supercharge.

Recently, inspired by the BLG theory, a large class of $\mathcal{N} \geq 4$ Chern-Simons theories (with ordinary Lie algebra gauge symmetry) has been constructed [28, 29] and their relation to string/$\mathcal{M}$-theory has been elucidated. It would be interesting to consider twisting those theories. As they include both Chern-Simons terms as well as non-linear sigma model with hyperKähler target space, they may reveal interesting connection between the pure Chern-Simons theory [18] and the Rozansky-Witten theory [19].

## Acknowledgments

KL thanks Seungjoon Hyun for encouragement. The work of KL is supported by the Basic Research Program of the Korea Science and Engineering Foundation under grant number R01-2004-000-10651-0. SL is supported in part by the KOSEF Grant R01-2006-000-10965-0 and the Korea Research Foundation Grant KRF-2007-331-C00073. The research of JHP is supported in part by the Center for Quantum Spacetime of Sogang University with grant number R11 - 2005 - 021, and by the Korea Science and Engineering Foundation grant funded by the Korea government (R01-2007-000-20062-0).

## A. Some useful relations

In curved backgrounds, the covariant derivative satisfies

$$[D_\mu, D_\nu]X^\nu - \tilde{F}_{\mu\nu}X^\nu + R_{\mu\nu}X^\nu = 0\,, \tag{A.1}$$

and hence

$$\begin{aligned}\mathrm{Tr}(D_\mu X_\nu, D^\mu X^\nu) =& \frac{1}{2}\mathrm{Tr}(D_\mu X_\nu - D_\nu X_\mu, D^\mu X^\nu - D^\nu X^\mu) + \mathrm{Tr}(D_\mu X^\mu, D_\mu X^\mu) \\ &+ X^\mu_a\left(\tilde{F}_{\mu\nu}{}^{ab}X^\nu_b - R_{\mu\nu}X^{\nu a}\right) + \partial_\mu \mathrm{Tr}(X^\nu, D_\nu X^\mu) - \partial_\mu \mathrm{Tr}(X^\mu, D_\nu X^\nu).\end{aligned} \tag{A.2}$$

With this and the decomposition of the bosonic fields (2.20), up to total derivatives, we can rewrite a bosonic part of the Bagger-Lambert-Gustavsson action for the twist:

$$\begin{aligned}
&\frac{1}{2}\mathrm{Tr}\big(D_\mu X_I, D^\mu X^I\big) + \frac{1}{12}\mathrm{Tr}\big([X^I,X^J,X^K],[X_I,X_J,X_K]\big) \\
&\quad\equiv \frac{1}{4}\mathrm{Tr}(D_\mu X_\nu - D_\nu X_\mu\,,\, D^\mu X^\nu - D^\nu X^\mu) + \frac{1}{2}\mathrm{Tr}(X^\mu,[D_\mu,D_\nu]X^\nu) \\
&\quad\quad +\frac{1}{2}\mathrm{Tr}\big(D_\mu X^\mu + i[X^1,X^2,X^3]\,,\, D_\mu X^\mu - i[X^1,X^2,X^3]\big) \\
&\quad\quad +\frac{1}{2}\mathrm{Tr}\bigg(D_\lambda Y^i + i\frac{1}{2}\epsilon_{\lambda\mu\nu}[X^\mu,X^\nu,Y^i], D^\lambda Y_i - i\frac{1}{2}\epsilon^{\lambda\rho\sigma}[X_\rho,X_\sigma,Y_i]\bigg) \\
&\quad\quad +\frac{1}{12}\mathrm{Tr}\Big([Y^i,Y^j,Y^k],[Y_i,Y_j,Y_k]\Big) + \frac{1}{4}\mathrm{Tr}\big([Y^i,Y^j,X^\mu],[Y_i,Y_j,X_\mu]\big)\,. \qquad \text{(A.3)}
\end{aligned}$$

Regarding the five-dimensional gamma matrices (2.14), two crucial Fierz identities follow from the completeness relations of $4\times 4$ symmetric and anti-symmetric matrices:

$$\begin{aligned}
4\delta_\alpha{}^\gamma\delta_\beta{}^\delta + 4\delta_\beta{}^\gamma\delta_\alpha{}^\delta + (C\gamma^{ij})_{\alpha\beta}(\gamma_{ij}C^{-1})^{\gamma\delta} &= 0\,, \\
2\delta_\alpha{}^\gamma\delta_\beta{}^\delta - 2\delta_\beta{}^\gamma\delta_\alpha{}^\delta + C_{\alpha\beta}C^{-1\gamma\delta} + (C\gamma^i)_{\alpha\beta}(\gamma_i C^{-1})^{\gamma\delta} &= 0\,. \qquad \text{(A.4)}
\end{aligned}$$

These further lead to other useful identities:

$$\begin{aligned}
(\gamma^i)^\alpha{}_\beta(\gamma_i)^\gamma{}_\delta &= 2\delta^\gamma{}_\beta\delta^\alpha{}_\delta - \delta^\alpha{}_\beta\delta^\gamma{}_\delta - 2C^{-1\alpha\gamma}C_{\beta\delta}\,, \qquad \text{(A.5)} \\
(\gamma^i)^\alpha{}_\beta(\gamma_{ji})^\gamma{}_\delta &= 2\delta^\alpha{}_\delta(\gamma_j)^\gamma{}_\beta - \delta^\alpha{}_\beta(\gamma_j)^\gamma{}_\delta - (\gamma_j)^\alpha{}_\beta\,\delta^\gamma{}_\delta - 2(\gamma_j C^{-1})^{\alpha\gamma}C_{\beta\delta}\,, \\
(\gamma^i)^\alpha{}_\beta(\gamma_{ijk})^\gamma{}_\delta &= 2\delta^\gamma{}_\beta(\gamma_{jk})^\alpha{}_\delta - \delta^\alpha{}_\beta(\gamma_{jk})^\gamma{}_\delta - 2C^{-1\alpha\gamma}(C\gamma_{jk})_{\beta\delta} - (\gamma_j)^\alpha{}_\beta(\gamma_k)^\gamma{}_\delta + (\gamma_k)^\alpha{}_\beta(\gamma_j)^\gamma{}_\delta\,,
\end{aligned}$$

$$\begin{aligned}
\delta^\beta{}_\gamma(\gamma_{ij})^\alpha{}_\delta + \delta^\alpha{}_\gamma(\gamma_{ij})^\beta{}_\delta - (\gamma_{ij})^\beta{}_\gamma\delta^\alpha{}_\delta - (\gamma_{ij})^\alpha{}_\gamma\delta^\beta{}_\delta + 2C_{\gamma\delta}(\gamma_{ij}C^{-1})^{\alpha\beta} \qquad \text{(A.6)} \\
+(\gamma_j)^\beta{}_\gamma(\gamma_i)^\alpha{}_\delta - (\gamma_i)^\beta{}_\gamma(\gamma_j)^\alpha{}_\delta + (\gamma_j)^\alpha{}_\gamma(\gamma_i)^\beta{}_\delta - (\gamma_i)^\alpha{}_\gamma(\gamma_j)^\beta{}_\delta = 0\,, \\
-\delta^\alpha{}_\beta(\gamma_j)^\gamma{}_\delta - (\gamma_j)^\alpha{}_\beta\delta^\gamma{}_\delta - (\gamma_j C^{-1})^{\alpha\gamma}C_{\beta\delta} - (C^{-1})^{\alpha\gamma}(C\gamma_j)_{\beta\delta} + \delta^\gamma{}_\beta(\gamma_j)^\alpha{}_\delta + (\gamma_j)^\gamma{}_\beta\delta^\alpha{}_\delta = 0\,.
\end{aligned}$$

## References


[1] R.C. Myers, *Dielectric-branes*, *JHEP* **12** (1999) 022 [hep-th/9910053].

[2] A. Basu and J.A. Harvey, *The M2-M5 brane system and a generalized Nahm's equation*, *Nucl. Phys.* **B 713** (2005) 136 [hep-th/0412310].

[3] D.-E. Diaconescu, *D-branes, monopoles and Nahm equations*, *Nucl. Phys.* **B 503** (1997) 220 [hep-th/9608163].

[4] D. Tsimpis, *Nahm equations and boundary conditions*, *Phys. Lett.* **B 433** (1998) 287 [hep-th/9804081].

[5] Z. Guralnik and S. Ramgoolam, *On the polarization of unstable D0-branes into non-commutative odd spheres*, *JHEP* **02** (2001) 032 [hep-th/0101001].

[6] J.H. Schwarz, *Superconformal Chern-Simons theories*, *JHEP* **11** (2004) 078 [hep-th/0411077].

[7] J. Bagger and N. Lambert, *Modeling multiple M2's*, *Phys. Rev.* **D 75** (2007) 045020 [hep-th/0611108]; *Gauge symmetry and supersymmetry of multiple M2-branes*, *Phys. Rev.* **D 77** (2008) 065008 [arXiv:0711.0955]; *Comments on multiple M2-branes*, *JHEP* **02** (2008) 105 [arXiv:0712.3738].

[8] A. Gustavsson, *Algebraic structures on parallel M2-branes*, arXiv:0709.1260.

[9] N. Lambert and D. Tong, *Membranes on an orbifold*, *Phys. Rev. Lett.* **101** (2008) 041602 [arXiv:0804.1114].

[10] J. Distler, S. Mukhi, C. Papageorgakis and M. Van Raamsdonk, *M2-branes on M-folds*, *JHEP* **05** (2008) 038 [arXiv:0804.1256].

[11] J. Gomis, G. Milanesi and J.G. Russo, *Bagger-Lambert theory for general Lie algebras*, *JHEP* **06** (2008) 075 [arXiv:0805.1012];
S. Benvenuti, D. Rodriguez-Gomez, E. Tonni and H. Verlinde, *$N = 8$ superconformal gauge theories and M2 branes*, arXiv:0805.1087;
P.-M. Ho, Y. Imamura and Y. Matsuo, *M2 to D2 revisited*, *JHEP* **07** (2008) 003 [arXiv:0805.1202];
P. De Medeiros, J.M. Figueroa-O'Farrill and E. Mendez-Escobar, *Lorentzian Lie 3-algebras and their Bagger-Lambert moduli space*, *JHEP* **07** (2008) 111 [arXiv:0805.4363];
J.M. Figueroa-O'Farrill, *Lorentzian Lie n-algebras*, arXiv:0805.4760;
M.A. Bandres, A.E. Lipstein and J.H. Schwarz, *Ghost-free superconformal action for multiple M2-branes*, *JHEP* **07** (2008) 117 [arXiv:0806.0054];
J. Gomis, D. Rodriguez-Gomez, M. Van Raamsdonk and H. Verlinde, *Supersymmetric Yang-Mills theory from Lorentzian three-algebras*, *JHEP* **08** (2008) 094 [arXiv:0806.0738];
B. Ezhuthachan, S. Mukhi and C. Papageorgakis, *D2 to D2*, *JHEP* **07** (2008) 041 [arXiv:0806.1639];
S. Cecotti and A. Sen, *Coulomb branch of the Lorentzian three algebra theory*, arXiv:0806.1990;
H. Verlinde, *D2 or M2? A note on membrane scattering*, arXiv:0807.2121.

[12] P.-M. Ho and Y. Matsuo, *M5 from M2*, *JHEP* **06** (2008) 105 [arXiv:0804.3629];
P.-M. Ho, Y. Imamura, Y. Matsuo and S. Shiba, *M5-brane in three-form flux and multiple M2-branes*, *JHEP* **08** (2008) 014 [arXiv:0805.2898].

[13] J.-H. Park and C. Sochichiu, *Single M5 to multiple M2: taking off the square root of Nambu-Goto action*, arXiv:0806.0335.

[14] I.A. Bandos and P.K. Townsend, *Light-cone M5 and multiple M2-branes*, arXiv:0806.4777.

[15] G. Bonelli, A. Tanzini and M. Zabzine, *Topological branes, p-algebras and generalized Nahm equations*, arXiv:0807.5113.

[16] N. Marcus, *The other topological twisting of $N = 4$ Yang-Mills*, *Nucl. Phys.* **B 452** (1995) 331 [hep-th/9506002].

[17] A. Kapustin and E. Witten, *Electric-magnetic duality and the geometric Langlands program*, hep-th/0604151.

[18] E. Witten, *Quantum field theory and the Jones polynomial*, *Commun. Math. Phys.* **121** (1989) 351.

[19] L. Rozansky and E. Witten, *Hyper-Kähler geometry and invariants of three-manifolds*, *Selecta Math.* **3** (1997) 401 [hep-th/9612216].

[20] M.A. Bandres, A.E. Lipstein and J.H. Schwarz, $N = 8$ *superconformal Chern-Simons theories*, *JHEP* **05** (2008) 025 [arXiv:0803.3242].

[21] J.-H. Park and D. Tsimpis, *Topological twisting of conformal supercharges*, *Nucl. Phys.* **B 776** (2007) 405 [hep-th/0610159].

[22] I. Jeon, J. Kim, N. Kim, S.-W. Kim and J.-H. Park, *Classification of the BPS states in Bagger-Lambert theory*, *JHEP* **07** (2008) 056 [arXiv:0805.3236].

[23] I. Jeon, J. Kim, B.-H. Lee, J.-H. Park and N. Kim, *M-brane bound states and the supersymmetry of BPS solutions in the Bagger-Lambert theory*, arXiv:0809.0856.

[24] C. Krishnan and C. Maccaferri, *Membranes on calibrations*, *JHEP* **07** (2008) 005 [arXiv:0805.3125].

[25] K. Hosomichi et al., *A nonperturbative test of $M2$-brane theory*, arXiv:0809.1771.

[26] P.S. Howe, G. Sierra and P.K. Townsend, *Supersymmetry in six-dimensions*, *Nucl. Phys.* **B 221** (1983) 331.

[27] J.-H. Park, *A study of two-/one-form superfields*, *Int. J. Mod. Phys.* **A 16** (2001) 1261 [math-ph/0103024].

[28] D. Gaiotto and E. Witten, *Janus configurations, Chern-Simons couplings and the theta-angle in $N = 4$ super Yang-Mills theory*, arXiv:0804.2907;
K. Hosomichi, K.-M. Lee, S. Lee, S. Lee and J. Park, $N = 4$ *superconformal Chern-Simons theories with hyper and twisted hyper multiplets*, *JHEP* **07** (2008) 091 [arXiv:0805.3662].

[29] H. Fuji, S. Terashima and M. Yamazaki, *A new $N = 4$ membrane action via orbifold*, arXiv:0805.1997;
O. Aharony, O. Bergman, D.L. Jafferis and J. Maldacena, $N = 6$ *superconformal Chern-Simons-matter theories, $M2$-branes and their gravity duals*, *JHEP* **10** (2008) 091 [arXiv:0806.1218];
M. Benna, I. Klebanov, T. Klose and M. Smedback, *Superconformal Chern-Simons theories and $AdS_4/CFT_3$ correspondence*, *JHEP* **09** (2008) 072 [arXiv:0806.1519];
Y. Imamura and K. Kimura, *Coulomb branch of generalized ABJM models*, arXiv:0806.3727;
K. Hosomichi, K.-M. Lee, S. Lee, S. Lee and J. Park, $N = 5, 6$ *superconformal Chern-Simons theories and $M2$-branes on orbifolds*, *JHEP* **09** (2008) 002 [arXiv:0806.4977];
S. Terashima and F. Yagi, *Orbifolding the membrane action*, arXiv:0807.0368;
J. Bagger and N. Lambert, *Three-algebras and $N = 6$ Chern-Simons gauge theories*, arXiv:0807.0163;
M. Schnabl and Y. Tachikawa, *Classification of $N = 6$ superconformal theories of ABJM type*, arXiv:0807.1102;
Y. Imamura and K. Kimura, $N = 4$ *Chern-Simons theories with auxiliary vector multiplets*, *JHEP* **10** (2008) 040 [arXiv:0807.2144];
O. Aharony, O. Bergman and D.L. Jafferis, *Fractional $M2$-branes*, arXiv:0807.4924.